Journal of Applied Mechanics and Technical Physics, Vol. 50, No. 6, pp. 998–1010, 2009

CALCULATION OF THE FORCE OF INTERACTION OF TWO DROPS IN A PLASTIC MEDIUM

Yu. V. Pivovarov

UDC 517.958.539.3

In calculating the force of interaction of two spherical drops, the stress tensor component normal to the drop surface is taken from the solution of the corresponding problem of the elasticity theory, while the shear component is determined by the plastic properties of the medium. The results of the calculations performed are demonstrated to be in good agreement with experimental data on the character of drop motion and on the yield point of the medium surrounding the drops.

Key words: *elasticity, plasticity, Mises yield condition.*

Introduction. Stebnovskii [1], following his previous study [2], considered the behavior of drops of liquid paraffin, sunflower-seed oil, and industrial oil in an alcohol–water solution of uniform density. He found that, if the distance between two drops is of the order of their sizes, they approach each other until they merge into one drop, independent of the system scale. The experimental setup used in [1] was insulated from external force and heat effects. It was found in those experiments that the approach is observed only if both drops possess surface tension.

Stebnovskii [3] also assumed that, at the initial stage of the process of drop approach, the ambient medium, which will be called the matrix as in [1], behaves as a solid that obeys Hooke's law. This assumption, however, does not explain the drop-approach mechanism, because the force acting on the drop from the side of the matrix is always zero by virtue of equations of equilibrium inside the drop and constant surface tension on the drop boundary. It is known from experiments, nevertheless, that water and, hence, the alcohol–water solution possess the yield point k_0 with the values in the interval from 10^{-4} to 10^{-3} Pa. The analysis performed in the present work shows that the absolute values of the shear stresses on the drop boundary cannot exceed the value of k_0, whereas there are no constraints on the normal stresses. Therefore, the shear stresses have to be corrected, while the normal stresses should be retained as they were in considering the matrix as an elastic medium. Then, if the force induced by the normal stresses is sufficiently high to overcome the medium resistance due to shear stresses, then the drop starts moving (in this case, the equations of equilibrium inside the drop become incompatible, and they should be replaced by the equations of dynamics of the drop considered as an elastic solid). The force accelerating the drop acts until the drop covers a certain distance of the order of one molecule size. (This conclusion is confirmed by the estimates obtained in the present work.) After that, the molecular bonds, which made the matrix acquire the properties of a solid, are broken, some volume around the drop becomes "liquid," and the drop continues to move owing to inertia, gradually decelerating owing to medium resistance until it stops completely. After a certain time, the matrix on a certain part on the drop surface again becomes "solid," which induces a force acting on the drop. The process is repeated again.

A method of calculating the force acting on the drop from the matrix at the initial stage of drop acceleration under the assumption that the entire volume occupied by the matrix is "solid" at the initial time (i.e., there are no "liquid" zones) is proposed in the present paper.

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; pivov@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 50, No. 6, pp. 100–114, November–December, 2009. Original article submitted June 9, 2008; revision submitted December 8, 2008.

1. Determining the Fields of Stresses and Displacements in an Elastic Medium Containing Two Drops Possessing Surface Tension. The first stage of calculating the force acting on the drop is solving the following problem of the elasticity theory: we have to find the components $T^k_{\alpha\beta}$, $\alpha\beta \in \{(RR), (R\theta), (\theta\theta), (\varphi\varphi)\}$ of the stress tensors T^k and the components u^k_α ($\alpha \in \{R, \theta\}$) of the displacement vectors $\boldsymbol{u}^k$ (the values $k = 1$ and 3 refer to the first and second drops, and the value $k = 2$ refers to the matrix) that satisfy the equations of the elasticity theory and the following boundary conditions on the surface of the first drop:

$$
\begin{gathered}
T^2_{RR} - T^1_{RR} = 2\gamma/R_0, \qquad T^2_{R\theta} - T^1_{R\theta} = 0, \\
u^2_R - u^1_R = 0, \qquad u^2_\theta - u^1_\theta = 0, \qquad \theta \in [0, \pi], \quad R = R_0.
\end{gathered} \tag{1}
$$

Similar conditions are imposed on the surface of the second drop. In Eqs. (1), R, θ, φ ($R \in [0, \infty)$, $\theta \in [0, \pi]$, and $\varphi \in [0, 2\pi)$) is the local spherical coordinate system in the drop (the angle θ is counted from the axis directed from the center of this drop to the center of the other drop), and R_0 and γ are the drop radius and its surface tension (the values of R_0 and γ are assumed to be identical for both drops). An axisymmetric case is considered. The first condition in system (1) defines the jump of the normal stresses proportional to the surface tension of the drop, as it is given in the classical hydrodynamics. The second condition states the continuity of the shear stresses; the third and fourth conditions define the continuity of the normal and shear displacements on the interface between two media.

Let us construct an algorithm for determining the tensors T^k and the vectors $\boldsymbol{u}^k$. For this purpose, we use the alternating Schwarz method, which consists in reducing the problem in the non-canonical domain to a sequence of problems in the canonical domains (external and internal domains of a sphere). The system of two drops is replaced by an equivalent system consisting of the drop and the mirror plane passing through the middle of the segment connecting the drop centers, orthogonal to this segment. This replacement is rather convenient, because it allows only one drop to be considered. The fields of stresses and displacements generated by the other drop correspond to the fields generated by the drop under consideration and reflected from the mirror surface.

The algorithm of the alternating Schwarz method can be formally presented as the calculation of the sums of the series:

$$
\begin{gathered}
T^k_{\alpha\beta} = \sum_{\nu=1}^{\infty} (T^{k,\nu}_{\alpha\beta} + T^{k,\nu}_{\alpha\beta\ \mathrm{ref}}), \qquad \alpha\beta \in \{(RR), (R\theta), (\theta\theta), (\varphi\varphi)\}, \\
u^k_\alpha = \sum_{\nu=1}^{\infty} (u^{k,\nu}_\alpha + u^{k,\nu}_{\alpha\ \mathrm{ref}}), \qquad \alpha \in \{R, \theta\}.
\end{gathered} \tag{2}
$$

The set of the first terms in brackets in Eqs. (2) will be called the direct solution at the νth iteration, and the set of the second terms will be called the reflected solution.

Let K_k, G_k, and m_k denote the coefficients of volume compression, shear moduli, and Poisson's numbers (inverse Poisson's ratios) of the drop material ($k = 1$) and the matrix material ($k = 2$).

The initial approximation, i.e., the direct solution at $\nu = 1$, is constructed as the solution of the problem of the elasticity theory with the boundary conditions (1) for one drop in an infinite ambient medium. The general solution of the equations of the elasticity theory, which is independent of θ, has the form [4]

$$
\begin{gathered}
T^{1,1}_{RR} = T^{1,1}_{\theta\theta} = T^{1,1}_{\varphi\varphi} = -4G_1 A^1_0 \,\frac{m_1 + 1}{m_1}, \qquad T^{1,1}_{R\theta} = 0, \\
u^{1,1}_R = -2A^1_0 \,\frac{m_1 - 2}{m_1}\, r, \qquad u^{1,1}_\theta = 0, \qquad T^{2,1}_{RR} = \frac{4G_2 D^1_0}{r^3}, \\
T^{2,1}_{\theta\theta} = T^{2,1}_{\varphi\varphi} = -\frac{2G_2 D^1_0}{r^3}, \qquad T^{2,1}_{R\theta} = 0, \qquad u^{2,1}_R = -\frac{D^1_0}{r^2}, \qquad u^{2,1}_\theta = 0,
\end{gathered} \tag{3}
$$

where $r = R/R_0$ and A^1_0 and D^1_0 are arbitrary constants. Substituting this solution into the boundary conditions (1), we obtain a system of equations for determining the constants A^1_0 and D^1_0:

$$
4G_2 D^1_0 + 4G_1 \,\frac{m_1 + 1}{m_1}\, A^1_0 = \frac{2\gamma}{R_0}, \qquad -D^1_0 + 2\,\frac{m_1 - 2}{m_1}\, A^1_0 = 0.
$$

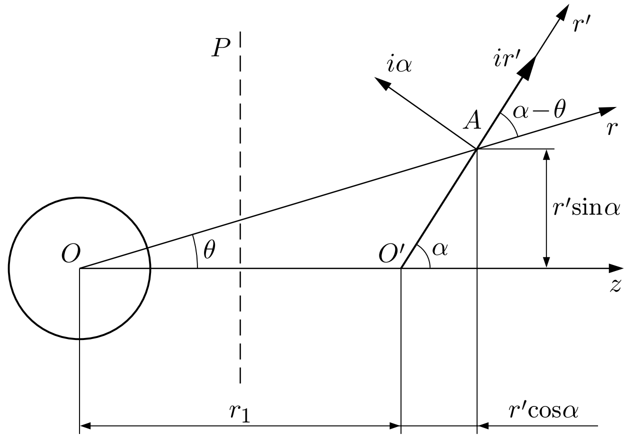

Fig. 1. Schematic of the problem and coordinate systems.

The solution of this system has the form

$$A_0^1 = \frac{\gamma}{2R_0} \frac{m_1}{2G_2(m_1-2)+G_1(m_1+1)}, \qquad D_0^1 = \frac{\gamma}{R_0 G_2} \frac{1}{2+G_1(m_1+1)/[G_2(m_1-2)]}. \tag{4}$$

Thus, the initial approximation is determined by Eqs. (3) and (4).

Let us transform the formula for D_0^1. Using the equality [5]

$$K_k = \frac{2G_k(m_k+1)}{3(m_k-2)} \qquad (k=1,2), \tag{5}$$

we obtain

$$D_0^1 = \frac{\gamma}{R_0 G_2} \frac{m_2-2}{2(m_2-2)+(K_1/K_2)(m_2+1)}.$$

Let the νth direct solution be known. We construct the νth reflected solution by introducing the following notation: O is the center of the drop image with respect to the mirror plane P, O' is the drop center, and A is an arbitrary point. Let r, θ, 0 be the spherical coordinates of the point A with respect to the point O, and let r', θ', 0 be the spherical coordinates of the point A with respect to the point O', r_1 is the distance between the drop centers and $\alpha = \pi - \theta'$. The drop radius R_0 is used as the measurement unit for the quantities r, r', and r_1. It follows from Fig. 1 that

$$r = (r_1^2 + 2r_1 r' \cos\alpha + r'^2)^{1/2}, \qquad \cos\theta = \frac{r_1 + r'\cos\alpha}{r}, \qquad \sin\theta = \frac{r'\sin\alpha}{r}.$$

In the system (r,θ,φ), the orthogonal unit vectors $\boldsymbol{i}_{r'}$, $\boldsymbol{i}_\alpha$, and $\boldsymbol{i}_\varphi$ of the coordinate directions r', α, and φ have the following coordinates:

$$\boldsymbol{i}_{r'} = (\cos\alpha', \sin\alpha', 0), \qquad \boldsymbol{i}_\alpha = (-\sin\alpha', \cos\alpha', 0), \qquad \boldsymbol{i}_\varphi = (0,0,1).$$

Here, we have $\alpha' = \alpha - \theta$. The stress tensor has the form

$$T = \begin{pmatrix} T_{RR} & T_{R\theta} & 0 \\ T_{R\theta} & T_{\theta\theta} & 0 \\ 0 & 0 & T_{\varphi\varphi} \end{pmatrix}.$$

The components of this tensor in the coordinate system (r',α,φ) are calculated by the formulas

$$T_{R'R'} = \boldsymbol{i}_{r'} \cdot T \cdot \boldsymbol{i}_{r'}, \quad T_{R'\alpha} = \boldsymbol{i}_{r'} \cdot T \cdot \boldsymbol{i}_\alpha, \quad T_{\alpha\alpha} = \boldsymbol{i}_\alpha \cdot T \cdot \boldsymbol{i}_\alpha, \quad T_{\varphi\varphi} = \boldsymbol{i}_\varphi \cdot T \cdot \boldsymbol{i}_\varphi. \tag{6}$$

The orthogonal unit vectors $\boldsymbol{i}_r$ and $\boldsymbol{i}_\theta$ of the directions r and θ have the coordinates $\boldsymbol{i}_r = (1,0,0)$ and $\boldsymbol{i}_\theta = (0,1,0)$. In the coordinate system (r',α,φ), the displacement vector components are written as

$$u_{r'} = u_r \cdot \boldsymbol{i}_{r'} \cdot \boldsymbol{i}_r + u_\theta \cdot \boldsymbol{i}_{r'} \cdot \boldsymbol{i}_\theta, \qquad u_\alpha = u_r \cdot \boldsymbol{i}_\alpha \cdot \boldsymbol{i}_r + u_\theta \cdot \boldsymbol{i}_\alpha \cdot \boldsymbol{i}_\theta, \tag{7}$$

thereby,

$$\cos\alpha' = \frac{r' + r_1\cos\alpha}{r}, \qquad \sin\alpha' = \frac{r_1\sin\alpha}{r}. \tag{8}$$

Substituting Eqs. (8) into Eqs. (6) and (7) and using the replacement $\theta' = \pi - \alpha$, we obtain the sought formulas for calculating the reflected solution:

$$\begin{gathered}
T^{k,\nu}_{RR\ \mathrm{ref}} = T^{4-k,\nu}_{RR} - (T^{4-k,\nu}_{RR} - T^{4-k,\nu}_{\theta\theta})r_1^2\sin^2(\theta'/r^2) + 2T^{4-k,\nu}_{R\theta}r_1\sin\theta'(r' - r_1\cos\theta')/r^2,\\
T^{k,\nu}_{\theta\theta\ \mathrm{ref}} = T^{4-k,\nu}_{\theta\theta} + (T^{4-k,\nu}_{RR} - T^{4-k,\nu}_{\theta\theta})r_1^2\sin^2(\theta'/r^2) - 2T^{4-k,\nu}_{R\theta}r_1\sin\theta'(r' - r_1\cos\theta')/r^2,\\
T^{k,\nu}_{\varphi\varphi\ \mathrm{ref}} = T^{4-k,\nu}_{\varphi\varphi},\\
T^{k,\nu}_{R\theta\ \mathrm{ref}} = (T^{4-k,\nu}_{RR} - T^{4-k,\nu}_{\theta\theta})r_1\sin\theta'(r' - r_1\cos\theta')/r^2 - T^{4-k,\nu}_{R\theta}(1 - 2r_1^2\sin^2(\theta'/r^2)),\\
u^{k,\nu}_{R\ \mathrm{ref}} = (u^{4-k,\nu}_{R}(r' - r_1\cos\theta') + u^{4-k,\nu}_{\theta}r_1\sin\theta')/r,\\
u^{k,\nu}_{\theta\ \mathrm{ref}} = (u^{4-k,\nu}_{R}r_1\sin\theta' - u^{4-k,\nu}_{\theta}(r' - r_1\cos\theta'))/r, \qquad k = 1, 2.
\end{gathered} \tag{9}$$

The arguments of the left sides of Eqs. (9) are the variables r' and θ', and the arguments of the right sides of these equations are the variables r and θ:

$$r = (r_1^2 - 2r_1r'\cos\theta' + r'^2)^{1/2}, \qquad \theta = \arccos[(r_1 - r'\cos\theta')/r].$$

It should be noted that the superscript in the right sides of Eqs. (9) at $k = 1$ has the value $k' = 4 - k = 3$. We put this superscript into correspondence to functions derived from the formulas of the external solution for the drop and having the constants G_1 and m_1 of the material of the other drop.

Thus, the reflected solution is constructed.

Let us derive formulas for calculating the $(\nu + 1)$th direct solution on the basis of the νth reflected solution. For this purpose, we require the sum of the νth reflected solution and the $(\nu + 1)$th direct solution to satisfy the homogeneous conditions (1):

$$\begin{gathered}
T^{2,\nu+1}_{RR} - T^{1,\nu+1}_{RR} = T^{1,\nu}_{RR\ \mathrm{ref}} - T^{2,\nu}_{RR\ \mathrm{ref}}, \qquad T^{2,\nu+1}_{R\theta} - T^{1,\nu+1}_{R\theta} = T^{1,\nu}_{R\theta\ \mathrm{ref}} - T^{2,\nu}_{R\theta\ \mathrm{ref}},\\
u^{2,\nu+1}_{R} - u^{1,\nu+1}_{R} = u^{1,\nu}_{R\ \mathrm{ref}} - u^{2,\nu}_{R\ \mathrm{ref}},\\
u^{2,\nu+1}_{\theta} - u^{1,\nu+1}_{\theta} = u^{1,\nu}_{\theta\ \mathrm{ref}} - u^{2,\nu}_{\theta\ \mathrm{ref}}, \qquad r = 1, \quad \theta \in [0, \pi]
\end{gathered} \tag{10}$$

(the primes at the variables r and θ of the reflected functions are omitted).

The right sides of the first and third equations of system (10) are expanded with respect to the Legendre polynomials $P_n(\cos\theta)$, and the right sides of the second and fourth equations are expanded with respect to the functions $dP_n(\cos\theta)/d\theta$ [4]:

$$\begin{gathered}
T^{1,\nu}_{RR\ \mathrm{ref}} - T^{2,\nu}_{RR\ \mathrm{ref}} = \sum_{n=0}^{\infty}\sigma^{\nu}_{n}P_n(\cos\theta), \qquad T^{1,\nu}_{R\theta\ \mathrm{ref}} - T^{2,\nu}_{R\theta\ \mathrm{ref}} = \sum_{n=1}^{\infty}\tau^{\nu}_{n}\frac{dP_n(\cos\theta)}{d\theta},\\
u^{1,\nu}_{R\ \mathrm{ref}} - u^{2,\nu}_{R\ \mathrm{ref}} = -\sum_{n=0}^{\infty}\xi^{\nu}_{n}P_n(\cos\theta), \qquad u^{1,\nu}_{\theta\ \mathrm{ref}} - u^{2,\nu}_{\theta\ \mathrm{ref}} = -\sum_{n=1}^{\infty}\eta^{\nu}_{n}\frac{dP_n(\cos\theta)}{d\theta}.
\end{gathered} \tag{11}$$

The expansion coefficients are calculated by the formulas

$$\sigma^{\nu}_{n} = \frac{2n+1}{2}\int_0^{\pi}(T^{1,\nu}_{RR\ \mathrm{ref}} - T^{2,\nu}_{RR\ \mathrm{ref}})P_n(\cos\theta)\sin\theta\,d\theta,$$

$$\tau^{\nu}_{n} = \frac{2n+1}{2n(n+1)}\int_0^{\pi}(T^{1,\nu}_{R\theta\ \mathrm{ref}} - T^{2,\nu}_{R\theta\ \mathrm{ref}})\frac{dP_n(\cos\theta)}{d\theta}\sin\theta\,d\theta,$$

$$\xi_n^\nu = -\frac{2n+1}{2}\int\limits_0^\pi (u^{1,\nu}_{R\ \mathrm{ref}} - u^{2,\nu}_{R\ \mathrm{ref}})P_n(\cos\theta)\sin\theta\,d\theta, \tag{12}$$

$$\eta_n^\nu = -\frac{2n+1}{2n(n+1)}\int\limits_0^\pi (u^{1,\nu}_{\theta\ \mathrm{ref}} - u^{2,\nu}_{\theta\ \mathrm{ref}})\,\frac{dP_n(\cos\theta)}{d\theta}\,\sin\theta\,d\theta.$$

Each of the functions that are united into a set determining the direct solution is also expanded into series with an arbitrary value of r [4]:

$$\begin{gathered}
T^{k,\nu+1}_{RR}(r,\theta) = \sum_{n=0}^{\infty} T^{k,\nu+1}_{RRn}(r)P_n(\cos\theta), \qquad T^{k,\nu+1}_{R\theta}(r,\theta) = \sum_{n=1}^{\infty} T^{k,\nu+1}_{R\theta n}(r)\,\frac{dP_n(\cos\theta)}{d\theta},\\
u^{k,\nu+1}_{R}(r,\theta) = \sum_{n=0}^{\infty} u^{k,\nu+1}_{Rn}(r)P_n(\cos\theta), \qquad u^{k,\nu+1}_{\theta}(r,\theta) = \sum_{n=1}^{\infty} u^{k,\nu+1}_{\theta n}(r)\,\frac{dP_n(\cos\theta)}{d\theta},\\
T^{k,\nu+1}_{\theta\theta}(r,\theta) = \sum_{n=0}^{\infty} T^{k,\nu+1}_{\theta\theta 1n}(r)P_n(\cos\theta) + \sum_{n=1}^{\infty} T^{k,\nu+1}_{\theta\theta 2n}(r)\,\frac{dP_n(\cos\theta)}{d\theta}\,\cot\,\theta,\\
T^{k,\nu+1}_{\varphi\varphi}(r,\theta) = \sum_{n=0}^{\infty} T^{k,\nu+1}_{\varphi\varphi 1n}(r)P_n(\cos\theta) + \sum_{n=1}^{\infty} T^{k,\nu+1}_{\varphi\varphi 2n}(r)\,\frac{dP_n(\cos\theta)}{d\theta}\,\cot\,\theta, \qquad k=1,2,3.
\end{gathered} \tag{13}$$

The functions at the summation sign are partial solutions of the equations of the elasticity theory if the following equalities are valid [4]:

$$\begin{gathered}
T^{1,\nu+1}_{RRn}(r) = 2G_1(A^{\nu+1}_n(n+1)(n^2-n-2-2/m_1)r^n + B^{\nu+1}_n n(n-1)r^{n-2}),\\
T^{1,\nu+1}_{R\theta n}(r) = 2G_1(A^{\nu+1}_n(n^2+2n-1+2/m_1)r^n + B^{\nu+1}_n(n-1)r^{n-2}),\\
u^{1,\nu+1}_{Rn}(r) = A^{\nu+1}_n(n+1)(n-2+4/m_1)r^{n+1} + B^{\nu+1}_n n r^{n-1},\\
u^{1,\nu+1}_{\theta n}(r) = A^{\nu+1}_n(n+5-4/m_1)r^{n+1} + B^{\nu+1}_n r^{n-1},\\
T^{1,\nu+1}_{\theta\theta 1n}(r) = -2G_1(A^{\nu+1}_n(n^2+4n+2+2/m_1)(n+1)r^n + B^{\nu+1}_n n^2 r^{n-2}),\\
T^{1,\nu+1}_{\theta\theta 2n}(r) = -2G_1(A^{\nu+1}_n(n+5-4/m_1)r^n + B^{\nu+1}_n r^{n-2}),\\
T^{1,\nu+1}_{\varphi\varphi 1n}(r) = 2G_1(A^{\nu+1}_n(n-2-2/m_1-4n/m_1)(n+1)r^n + B^{\nu+1}_n n r^{n-2}),\\
T^{1,\nu+1}_{\varphi\varphi 2n}(r) = 2G_1(A^{\nu+1}_n(n+5-4/m_1)r^n + B^{\nu+1}_n r^{n-2}),\\
T^{2,\nu+1}_{RRn}(r) = 2G_2(-C^{\nu+1}_n n(n^2+3n-2/m_2)/r^{n+1} + D^{\nu+1}_n(n+1)(n+2)/r^{n+3}),\\
T^{2,\nu+1}_{R\theta n}(r) = 2G_2(C^{\nu+1}_n(n^2-2+2/m_2)/r^{n+1} - D^{\nu+1}_n(n+2)/r^{n+3}),\\
u^{2,\nu+1}_{Rn}(r) = C^{\nu+1}_n n(n+3-4/m_2)/r^n - D^{\nu+1}_n(n+1)/r^{n+2},\\
u^{2,\nu+1}_{\theta n}(r) = C^{\nu+1}_n(-n+4-4/m_2)/r^n + D^{\nu+1}_n/r^{n+2},\\
T^{2,\nu+1}_{\theta\theta 1n}(r) = 2G_2(C^{\nu+1}_n n(n^2-2n-1+2/m_2)/r^{n+1} - D^{\nu+1}_n(n+1)^2/r^{n+3}),\\
T^{2,\nu+1}_{\theta\theta 2n}(r) = -2G_2(C^{\nu+1}_n(-n+4-4/m_2)/r^{n+1} + D^{\nu+1}_n/r^{n+3}),\\
T^{2,\nu+1}_{\varphi\varphi 1n}(r) = 2G_2(C^{\nu+1}_n n(n+3-4n/m_2-2/m_2)/r^{n+1} - D^{\nu+1}_n(n+1)/r^{n+3}),\\
T^{2,\nu+1}_{\varphi\varphi 2n}(r) = 2G_2(C^{\nu+1}_n(-n+4-4/m_2)/r^{n+1} + D^{\nu+1}_n/r^{n+3}).
\end{gathered} \tag{14}$$

At $k = 3$, the functions $T^{k,\nu+1}_{RRn}(r), \ldots, T^{k,\nu+1}_{\varphi\varphi 2n}(r)$ are constructed by the same formulas as at $k = 2$, but the constants G_2 and m_2 in these formulas should be replaced by the constants G_1 and m_1, respectively.

Substituting Eqs. (11), (13), and (14) into Eqs. (10), we obtain the following systems of equations for determining the constants $A^{\nu+1}_n$, $B^{\nu+1}_n$, $C^{\nu+1}_n$, and $D^{\nu+1}_n$:

$$\begin{gathered}-4G_1(m_1+1)A^{\nu+1}_0/m_1 + 4G_2 D^{\nu+1}_0 = \sigma^\nu_0,\\ 2(m_1-2)A^{\nu+1}_0/m_1 - D^{\nu+1}_0 = -\xi^\nu_0, \qquad \nu = \overline{1,\infty};\end{gathered} \tag{15}$$

$$\begin{gathered}-2G_1(n+1)(n^2-n-2-2/m_1)A^{\nu+1}_n - 2G_1 n(n-1)B^{\nu+1}_n\\ -\,2G_2 n(n^2+3n-2/m_2)C^{\nu+1}_n + 2G_2(n+1)(n+2)D^{\nu+1}_n = \sigma^\nu_n,\\ -2G_1(n^2+2n-1+2/m_1)A^{\nu+1}_n - 2G_1(n-1)B^{\nu+1}_n + 2G_2(n^2-2+2/m_2)C^{\nu+1}_n - 2G_2(n+2)D^{\nu+1}_n = \tau^\nu_n,\\ -(n+1)(n-2+4/m_1)A^{\nu+1}_n - nB^{\nu+1}_n + n(n+3-4/m_2)C^{\nu+1}_n - (n+1)D^{\nu+1}_n = -\xi^\nu_n,\\ -(n+5-4/m_1)A^{\nu+1}_n - B^{\nu+1}_n + (-n+4-4/m_2)C^{\nu+1}_n + D^{\nu+1}_n = -\eta^\nu_n,\\ n = \overline{1,\infty}, \qquad \nu = \overline{1,\infty}.\end{gathered} \tag{16}$$

When these constants are determined from systems (15) and (16), the direct solution for the $(\nu+1)$th iteration is constructed by Eqs. (13) and (14).

At $k = 1$ and $k' = 4 - k = 3$, the functions in the right sides of Eqs. (9) should be calculated by Eqs. (3), (4), (13), and (14) with $k = 2$, but the constants G_2 and m_2 in Eqs. (3) and (14) should be replaced by the constants G_1 and m_1, because we consider the fields of stresses and displacements, which are external to the drop and have the shear modulus and Poisson's number of the material of the other drop.

Thus, the algorithm for calculating the tensors T^k and the vectors $\boldsymbol{u}^k$ ($k = 1$ and 2) is constructed.

2. Analytical Approximation. It is shown below that the z-projection of the force acting on the drop at the initial stage of its acceleration (see Fig. 1) is

$$F'_z = F_{z1} + \pi^2 R_0^2 k_0,$$

where

$$F_{z1} = -2\pi R_0^2 \int\limits_0^\pi T^2_{RR}\cos\theta\sin\theta\, d\theta \tag{17}$$

at $R = R_0$ is the z-projection of the force generated by the normal stresses on the drop surface and k_0 is the positive shear stress equal to the yield point of the matrix. It is known from the experiment that the force F'_z at $r_1 < 3.2$ is negative, i.e., the drops are attracted to each other. Let us estimate the quantity F_{z1} analytically. For this purpose, we put series (2) into correspondence to the series

$$F_{z1} = \sum_{\nu=1}^\infty \sum_{n=0}^\infty F^\nu_{z1n},$$

where the subscript n indicates the number of the harmonic in the expansion of the solution with respect to the Legendre polynomials, and calculate the highest term of this series.

Substituting Eq. (3) into Eq. (9) with $k = 2$, we obtain

$$\begin{gathered}T^{2,1}_{RR\ \mathrm{ref}}(1,\theta) = 2D^1_0 G_2\Big(2 - \frac{3r_1^2\sin^2\theta}{r_1^2+1-2r_1\cos\theta}\Big)\frac{1}{(r_1^2+1-2r_1\cos\theta)^{3/2}},\\ T^{2,1}_{R\theta\ \mathrm{ref}}(1,\theta) = 6D^1_0 G_2\Big(\frac{3r_1^2\sin^2\theta}{r_1^2+1-2r_1\cos\theta} - 1\Big)\frac{1}{(r_1^2+1-2r_1\cos\theta)^{3/2}}.\end{gathered} \tag{18}$$

For this solution, the integral in Eq. (17) is equal to zero. Therefore, this approximation does not yield the force of attraction of the drops. In the next approximation, we have

$$T^{1,1}_{RR\ \mathrm{ref}} - T^{2,1}_{RR\ \mathrm{ref}} = (G_1/G_2 - 1)T^{2,1}_{RR\ \mathrm{ref}}, \qquad T^{1,1}_{R\theta\ \mathrm{ref}} - T^{2,1}_{R\theta\ \mathrm{ref}} = (G_1/G_2 - 1)T^{2,1}_{R\theta\ \mathrm{ref}},$$

$$u^{1,1}_{R\ \mathrm{ref}} - u^{2,1}_{R\ \mathrm{ref}} = 0, \qquad u^{1,1}_{\theta\ \mathrm{ref}} - u^{2,1}_{\theta\ \mathrm{ref}} = 0. \tag{19}$$

Substituting Eqs. (18) and (19) into Eq. (12), we obtain

$$\sigma^1_0 = \xi^1_0 = \sigma^1_1 = \tau^1_1 = \xi^1_1 = \eta^1_1 = 0.$$

Therefore, we have

$$A^2_0 = D^2_0 = A^2_1 = B^2_1 = C^2_1 = D^2_1 = 0.$$

Then, we calculate the values

$$\sigma^1_2 = 2D^1_0 G_2(G_1/G_2 - 1)r_1^{-3}, \qquad \tau^1_2 = D^1_0 G_2(G_1/G_2 - 1)r_1^{-3}, \qquad \xi^1_2 = \eta^1_2 = 0.$$

Substituting these values into system (16) and solving this system, we find

$$C^2_2 = -\frac{5}{4}\,\frac{m_2\tau^1_2}{(7G_2 + 8G_1)m_2 - 5(G_2 + 2G_1)}, \quad D^2_2 = \frac{6C^2_2}{5}, \quad A^2_2 = 0, \quad B^2_2 = \Big(\frac{16}{5} - \frac{4}{m_2}\Big)C^2_2.$$

Using formulas (14), (13), (9), and (17), we obtain

$$F^a_{z1} = -\frac{48\pi R_0\gamma(G_1/G_2 - 1)(m_2 - 2)(m_2 + 1)r_1^{-7}}{(2m_2 - 4 + (K_1/K_2)(m_2 + 1))((7 + 8G_1/G_2)m_2 - 5 - 10G_1/G_2)}. \tag{20}$$

The quantity F^a_{z1}, responsible for a certain force of attraction of the drops at $G_1/G_2 > 1$ and a sufficiently small value of r_1, yields information about the sign of F_{z1}.

3. Calculating the Force Acting on the Drop. The matrix material is assumed to be weakly compressible (the coefficient of volume compression is $K_2 = 2\cdot 10^9$ Pa). The values of the second invariants of the strain rate tensor and the stress tensor deviator, as well as the form of the matrix continuity equation are determined below. As the allowance for matrix compressibility yields negligibly small corrections to the values of the corresponding quantities, we assume that the matrix is incompressible for simplicity. (In determining the fields of stresses and displacements in Sec. 1, the assumption about the matrix material compressibility was essential, because these fields are identically equal to zero for an incompressible matrix material.)

The stress tensors T^1 for the drop and T^2 for the matrix on the drop surface S are related by the condition [see Eq. (1)]

$$T^2\cdot\boldsymbol{n} = (T^1 + (2\gamma/R_0)I)\cdot\boldsymbol{n},$$

where I is the unit tensor and $\boldsymbol{n}$ is the external normal to the surface S. The elasticity theory predicts that the force acting on the drop is

$$\boldsymbol{F} = \iint\limits_S T^2\cdot\boldsymbol{n}\,dS = \iint\limits_S \Big(T^1 + I\,\frac{2\gamma}{R_0}\Big)\cdot\boldsymbol{n}\,dS = \iiint\limits_V \Big(\operatorname{div} T^1 + \frac{2\gamma}{R_0}\operatorname{div} I\Big)\,dV = 0, \tag{21}$$

because

$$\operatorname{div} T^1 = 0$$

by virtue of the equilibrium equations; the divergence of the unit tensor I is also equal to zero. In Eq. (21), V is the volume occupied by the drop. The transition from the surface integral to the volume integral is performed by the Gauss–Ostrogradskii formula.

Let us consider equality (21) in a cylindrical coordinate system $(\tilde r, \varphi, z)$ related to the spherical coordinate system (R, θ, φ) introduced in Sec. 1:

$$\tilde r = R\sin\theta, \qquad z = -R\cos\theta.$$

Then, we obtain

$$\boldsymbol{F} = (0, 0, F_z),$$

where

$$F_z = F_{z1} + F_{z2} = 0, \tag{22}$$

F_{z1} and F_{z2} are the z-projections of the integral of the normal and shear stresses, respectively, over the drop surface:

$$F_{z1} = -2\pi R_0^2 \int_0^\pi T_{RR}^2 \cos\theta \sin\theta \, d\theta, \qquad F_{z2} = 2\pi R_0^2 \int_0^\pi T_{R\theta}^2 \sin^2\theta \, d\theta, \qquad R = R_0. \tag{23}$$

Let the medium be plastic, i.e., possess a certain yield point k_0. The simplest model of such a medium is Bingham's model [6, 7]. As previously, we assume that the drop is an elastic solid if its yield point is rather high. In such a model, the domain occupied by the matrix is divided into two parts: "liquid" subdomain where the value of $|D|$ is rigorously greater than zero ($|D|^2 = \sum\limits_{i,j} D_{ij}^2$; D is the strain rate tensor) and "solid" subdomain where $|D| \equiv 0$. In this case, the stress tensor has the form

$$P = -pI + s,$$

where p is the hydrodynamic pressure, I is the unit tensor, and s is the deviator of the tensor P. The following relation between s and D is postulated:

$$s_{ij} = (2\rho_0\nu_0 + \sqrt{2}\, k_0/|D|) D_{ij} \tag{24}$$

(ρ_0 and ν_0 are the density and kinematic viscosity of the matrix material). Raising Eq. (24) to the second power, performing summation with respect to the subscripts i and j, and canceling the necessary terms at $|D| \to 0$, we obtain the Mises yield condition satisfied on the boundary of the "liquid" subdomain:

$$|s|^2 = 2k_0^2. \tag{25}$$

Here, we have

$$|s|^2 = \sum_{i,j} s_{ij}^2.$$

In the axisymmetric case, Eq. (25) in the spherical coordinates acquires the form

$$s_{RR}^2 + s_{\theta\theta}^2 + s_{\varphi\varphi}^2 + s_{R\theta}^2 + s_{\theta R}^2 = 2k_0^2. \tag{26}$$

Let the velocity field in the matrix be identically equal to zero, but condition (26) be satisfied on a certain part of the drop surface. As the diagonal components of the tensor s vanish in this case and the non-diagonal components are equal to each other by virtue of tensor symmetry, Eq. (26) has the form

$$|s_{R\theta}| = |P_{R\theta}| = k_0, \qquad \theta \in U, \quad R = R_0,$$

where U is a certain set.

According to Bingham's theory of the fluid, the stress tensor P in the "solid" subdomain can be undetermined, i.e., it can be a multivalued function [6]. The shear component of the tensor P is in the range

$$-k_0 \le P_{R\theta} \le k_0, \qquad \theta \in [0, \pi], \quad R = R_0.$$

The following question arises: How is it possible to obtain information about the properties of the function $P_{RR}(\theta)$ at $R = R_0$? The following answer is proposed: As Bingham's rheological model of the fluid includes an elastic element represented in rheological schemes by a spring [8], the normal stresses on the drop surface in a quiescent matrix should be obtained by solving the problem of the elasticity theory, which was considered in Sec. 1:

$$P_{RR} = T_{RR}^2, \qquad \theta \in [0, \pi], \quad R = R_0. \tag{27}$$

Let us assume that the drop starts to move. The strain rate tensor is not changed when we pass to the moving coordinate system; hence, the drop motion can be considered in a spherical coordinate system fitted to the drop center. In the axisymmetric case, we have the following non-zero components of the strain rate tensor [9]:

$$
\begin{gathered}
D_{RR} = \frac{\partial u}{\partial R}, \qquad D_{R\theta} = D_{\theta R} = \frac{\partial v}{\partial R} - \frac{v}{R} + \frac{1}{R}\frac{\partial u}{\partial \theta}, \\
D_{\theta\theta} = \frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{u}{R}, \qquad D_{\varphi\varphi} = \frac{v}{R}\cot\theta + \frac{u}{R}
\end{gathered}
\tag{28}
$$

(u and v are the velocity components in the directions R and θ, respectively).

Considering the problem in the drop-fitted coordinate system, we can conclude that the no-slip conditions should be satisfied on the drop surface:

$$
u = v = 0 \quad \text{at} \quad R = R_0, \qquad \theta \in [0, \pi]. \tag{29}
$$

These conditions and the continuity equation

$$
\frac{\partial u}{\partial R} + \frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{2u}{R} + \frac{v}{R}\cot\theta = 0
$$

yield

$$
\frac{\partial u}{\partial R} = 0, \qquad R = R_0, \quad \theta \in [0, \pi]. \tag{30}
$$

Substitution of Eqs. (29) and (30) into Eq. (28) yields the following equalities on the drop surface:

$$
D_{RR} = D_{\theta\theta} = D_{\varphi\varphi} = 0, \qquad D_{R\theta} = D_{\theta R} = \frac{\partial v}{\partial R}.
$$

Then, we have

$$
|D| = \Big(\Big|\sum_{i,j} D_{ij}^2\Big|\Big)^{1/2} = \sqrt{2}\,\Big|\frac{\partial v}{\partial R}\Big|,
$$

i.e., the tensor P has the components

$$
\begin{gathered}
P_{RR} = P_{\theta\theta} = P_{\varphi\varphi} = -p, \\
P_{R\theta} = P_{\theta R} = 2\rho_0\nu_0\frac{\partial v}{\partial R} + k_0 \operatorname{sign}\Big(\frac{\partial v}{\partial R}\Big) \equiv s_{R\theta}, \qquad R = R_0, \quad \theta \in [0, \pi].
\end{gathered}
\tag{31}
$$

Let us introduce the function $\xi(t)$ as the fraction of non-broken molecular bonds, which make the matrix acquire the properties of a solid, and assume that

$$
P_{RR} = \xi(t)T_{RR}^2 - (1 - \xi(t))p, \qquad \theta \in [0, \pi], \quad R = R_0.
$$

At the initial stage of drop acceleration, when the function $\xi(t)$ is close to unity, equality (27) is satisfied. We can demonstrate that we have $\operatorname{sign}(\partial v/\partial R) = -\operatorname{sign}(F_{z1})$ [the quantity F_{z1} is determined by Eq. (23)] is there are no reverse flow regions on the drop surface. Then, at the initial stage of drop acceleration, when the velocities are still rather low, equality (31) acquires the form

$$
P_{R\theta} = -k_0 \operatorname{sign}(F_{z1}).
$$

The force induced by shear stresses is

$$
F'_{z2} = 2\pi R_0^2 \int_0^\pi P_{R\theta}\sin^2\theta\, d\theta = -\pi^2 R_0^2 k_0 \operatorname{sign}(F_{z1}).
$$

Thus, if the drop starts to move, then the force acting on the drop is determined by the formula

$$
F'_z = F_{z1} - \pi^2 R_0^2 k_0 \operatorname{sign}(F_{z1}).
$$

The terms in the right side of this expression have different signs. If the first term dominates, then the drop is accelerated; if the second term dominates, then the drop is decelerated. In the latter case, minor deviations of the drop from the equilibrium conditions give rise to a force returning the drop to a position corresponding to the previous equilibrium state. Therefore, the drop can start to move only if the following inequality is valid:

$$|F_{z1}| > \pi^2 R_0^2 k_0.$$

Thus, the force F'_z acting on the drop at the initial stage of its acceleration is determined as

$$F'_z = \begin{cases} 0, & |F_{z1}| \le \pi^2 R_0^2 k_0, \\ F_{z1} - \pi^2 R_0^2 k_0 \operatorname{sign}(F_{z1}), & |F_{z1}| > \pi^2 R_0^2 k_0. \end{cases} \tag{32}$$

4. Results of Calculations. A computer code was written for implementing the algorithm described above. To test this code, we calculated the errors of satisfaction of difference analogs of the equilibrium equations in stresses and displacements at certain points of the inner and outer parts of the drop and on the drop boundary. For the step in terms of r equal to 0.0001 and the step in terms of θ equal to $\pi/10{,}000$, 30 harmonics, 20 iterations at $r_1 = 3.2$, $K_1 = 1/6 \cdot 10^{10}$ Pa, $K_2 = 2 \cdot 10^9$ Pa, $m_1 = 4$, $m_2 = 2.6$, $G_1 = 10^9$ Pa, $G_2 = 5 \cdot 10^8$ Pa, $R_0 = 0.005$ m, and $\gamma = 0.02$ N/m (the choice of these values of parameters is explained below), the errors of satisfaction of the equilibrium equations in stresses normalized to the maximum of the function $|T^2_{RR}|$ varied from 10^{-11} to 10^{-7}, and the errors of satisfaction of the equilibrium equations in displacements normalized to the maximum of the function $|u^2_R|$ varied from 10^{-10} to 10^{-5}. The greater values in the latter case could be attributed to the higher order of the equilibrium equations in displacements, as compared with the equilibrium equations in stresses. Integrals (12) were calculated by Simpson's rule with 2000 divisions of the segment $[0, \pi]$. The calculated errors of satisfaction of conditions (1), which were normalized to the maximums of the corresponding functions, were within the range from 10^{-11} to 10^{-9}. We also calculated the error of satisfaction of the condition of the zero sum of the integrals of the normal and shear stresses over the drop surface [see Eqs. (22) and (23)] and the error of calculating the value of F_{z1} with the use of two iterations and three harmonics (zeroth, first, and second), as compared with the error arising in calculations by Eq. (20). Both errors were equal to $5 \cdot 10^{-12}$.

To perform these calculations, we had to define the yield point of the matrix k_0, the coefficients of volume compression K_k, the shear moduli G_k, and Poisson's numbers (inverse to Poisson's ratios) $m_k \in [2, \infty)$ (with $m_k = 2$ corresponding to an incompressible medium) of the drop material ($k = 1$) and the matrix material ($k = 2$). These quantities are not independent because they are related by equalities (5).

Let us choose the initial data for the problem. The compressibility coefficients for water ($\chi_2 = 5 \cdot 10^{-5}$ atm^{-1}) and various oils were given in [10]. The compressibility coefficients for various oils at atmospheric pressure are close to $\chi_1 = 6 \cdot 10^{-5}$ atm^{-1}. Passing to the SI system of units, we use the formula $K_k = 1/\chi_k$ ($k = 1, 2$) and obtain $K_1 = 1/6 \cdot 10^{10}$ Pa and $K_2 = 2 \cdot 10^9$ Pa.

The following question arises: Which values should be set for the shear moduli and Poisson's numbers? Apakashev and Pavlov performed experiments to study the water flow decaying due to inertia in a cylindrical vessel and obtained the shear modulus equal to 10^{-6} Pa. Romanov and Sapozhnikov [12] performed experiments aimed at studying high-frequency (22 Hz and more) motion of a cylindrical shell made of steel foil around a motionless cylindrical skeleton with a fluid contained in a small gap between the skeleton and the shell and without this fluid; by comparing the amplitude–frequency characteristics of two processes, they obtained the shear moduli for water ($G_2 = 18$ Pa) and paraffin oil ($G_1 = 31$ Pa). Thus, the shear modulus of water substantially depends on the character of the process under study: as the frequency of oscillations is increased from a value close to zero to 22 Hz, the value of G_2 increases by more than seven orders. Note that the processes with strains of the order of unity were studied in both experiments. In the problem of approaching drops, the displacements due to surface tension are of the order of the size of one molecule, and the corresponding strains range from 10^{-8} to 10^{-7}, i.e., are extremely small. Even for $G_2 = 18$ Pa, Eq. (5) yields $m_2 - 2 \approx 10^{-8}$. Then, for $\gamma = 0.02$ N/m [3], $R_0 = 0.005$ m, $r_1 = 3.2$, and $G_1/G_2 = 2$ (approximate value for the data of [12] given above), Eq. (20) yields $F_{z1} \approx -2.5 \cdot 10^{-15}$ N. It follows from Eq. (32) that

$$F_{z1} + \pi^2 R_0^2 k_0 < 0.$$

It is known from the experiment that the drops become attracted to each other at $r_1 = 3.2$. Then, we obtain the estimate

$$k_0 < -F_{z1}/(\pi^2 R_0^2) \sim 10^{-11} \text{ Pa}.$$

Meanwhile, it is known from other experiments that k_0 lies in the range from 10^{-4} to 10^{-3} Pa. To explain experimental data, therefore, the value of $m_2 - 2$ should be of the order of unity. At such small strains, the water–alcohol solution behaves as a usual solid. These conclusions are also valid for the drop material. Poisson's numbers typical for solids are $1/\nu \geqslant 3$–4.

We use the above-listed values of r_1, R_0, γ, and G_1/G_2 as the initial data and denote the upper boundary of the values of k_0 at which $F'_z < 0$ by

$$k_{0\,\max} = -F_{z1}/(\pi^2 R_0^2). \tag{33}$$

Defining m_k, we can find the value of m_{3-k} from Eq. (5):

$$m_{3-k} = \Big(2\,\frac{G_k}{G_{3-k}}\,\frac{K_{3-k}}{K_k} - \frac{m_k - 2}{m_k + 1}\Big) \Big/ \Big(\frac{G_k}{G_{3-k}}\,\frac{K_{3-k}}{K_k} - \frac{m_k - 2}{m_k + 1}\Big), \qquad k = 1, 2. \tag{34}$$

If $G_k K_{3-k}/(G_{3-k} K_k) < 1$, the following condition should be satisfied for a positive value of m_{3-k} to exist:

$$m_k < m_{k\,\max}, \qquad m_{k\,\max} = \Big(2 + \frac{G_k}{G_{3-k}}\,\frac{K_{3-k}}{K_k}\Big) \Big/ \Big(1 - \frac{G_k}{G_{3-k}}\,\frac{K_{3-k}}{K_k}\Big).$$

Figure 2 shows the dependence of $k_{0\,\max}$ on m_2 obtained by calculations with the data indicated above. The value of $m_{2\,\max}$ is 29/7. The values of m_1 were calculated by Eq. (34). The calculation was performed from the value $m_2 = 2.1$. In the entire examined range of m_2, it is seen that the value of $k_{0\,\max}$ is within the range consistent with the experimental data (from 10^{-4} to 10^{-3} Pa), and the analytical approximation yields the value of $k_{0\,\max}$, which is smaller than the numerical value by less than a factor of 2. The error of calculating $k_{0\,\max}$ due to the neglect of terms with $\nu \geq 6$ is about 10^{-3}%.

In further calculations, we used $m_2 = 2.6$ (thereby, $m_1 = 4$, $k_{0\,\max} = 4.529 \cdot 10^{-4}$ Pa, $G_2 = 5 \cdot 10^8$ Pa, and $G_1 = 10^9$ Pa) and $k_0 = 0.0003$ Pa (mean geometric value between the minimum and maximum values).

Figure 3 shows the functions F_{z1}, F'_z, and F^a_{z1} [see Sec. 3 and Eq. (20)] versus r_1. At $r_1 \approx 3.36$, we have $F'_z = 0$, i.e., the drop cannot start moving at this value of r_1. With increasing r_1, the ratio F_{z1}/F^a_{z1} decreases from $F_{z1}/F^a_{z1} = 4$ at $r_1 = 2$ to $F_{z1}/F^a_{z1} = 1.47$ at $r_1 = 4$.

The effect of the ratio G_1/G_2 on the considered characteristics was studied. At $G_1/G_2 > 1$, we had $m_2 = 2.6$, and the value of m_1 was calculated by Eq. (34). At $G_1/G_2 < 1$ (in [12], such a ratio was obtained for hydraulic oil with $G_1 = 13$ Pa and water with $G_2 = 18$ Pa), we had $m_1 = 2.6$, and the value of m_2 was calculated by Eq. (34). In the first case, the following condition should be satisfied for the value of m_1 to be positive:

$$\frac{G_1}{G_2} < \frac{K_1}{K_2}\,\frac{m_2 + 1}{m_2 - 2} = 5.$$

In the second case, the following condition should be satisfied for m_2 to be positive:

$$\frac{G_1}{G_2} > \frac{K_1}{K_2}\,\frac{m_1 - 2}{m_1 + 1} = \frac{5}{36}.$$

Figure 4 shows the function $k_{0\,\max}$ [see Eq. (33)] determined numerically on the basis of the model developed in Sec. 1 (solid curve) and analytically by Eq. (20) (dot-and-dashed curve) versus the ratio $G_1/G_2 \in [1, 5)$ at $r_1 = 3.2$. For $G_1/G_2 \in [1.2,\ 4.5]$, the value of $k_{0\,\max}$ is seen to be in the range from 10^{-4} to 10^{-3} Pa corresponding to experimental data. The analytical values of $k_{0\,\max}$ are smaller than the values of $k_{0\,\max}$ obtained by numerical calculations by less than a factor of 2.

Figure 5 shows the functions F_{z1}, F'_z, and F^a_{z1} versus the ratio $G_1/G_2 \in [0.2,\ 5)$ at $r_1 = 3.2$, $k_0 = 0.0003$ Pa, and the values of m_1 and m_2 determined by the algorithm described above. The drops are seen to diverge at $G_1/G_2 < 0.5$ ($F'_z > 0$) and to approach each other at $G_1/G_2 > 1.7$ ($F'_z < 0$). At $0.5 < G_1/G_2 < 1.7$, the drops remain in the state at rest ($F'_z = 0$). Note, if F_{z1} is expanded into a series with respect to ν [see Eqs. (2)], the terms of this series have constant signs at $G_1/G_2 > 1$ and alternating signs at $G_1/G_2 < 1$. At $G_1/G_2 = 1$, we have $F_{z1} = 0$, and the exact solution of the problem is obtained in one iteration.

Let us estimate the path covered by the drop and the time of drop acceleration and deceleration in one cycle (see Introduction). It is known from the experiment that the mean velocity of the drop with a radius of 0.005 m at $r_1 = 3$ is $V_{\rm mean} = -3.68 \cdot 10^{-7}$ m/sec. The drop density is $\rho_0 = 10^3$ kg/m^3, and its mass is $m = (4\pi/3)\rho_0 R_0^3 = 5.24 \cdot 10^{-4}$ kg. The time needed for the drop to cover the distance Δz is $\Delta t = \sqrt{2\Delta z/a}$ ($a = F'_z/m = -2.18 \cdot 10^{-4}$ m/sec^2 is the drop acceleration). Then, its mean velocity is $V_{\rm mean} = \Delta z/\Delta t = -\sqrt{\Delta z a/2}$.

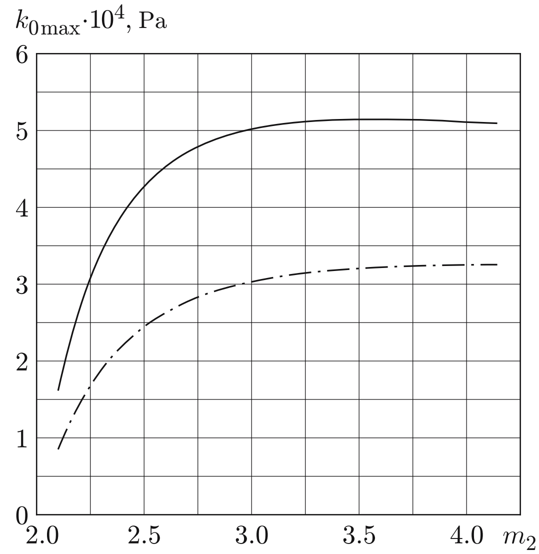

Fig. 2. Dependence of $k_{0\,\max}$ on m_2: the solid and dashed curves show the results calculated by the numerical model and by the approximate formula (20), respectively.

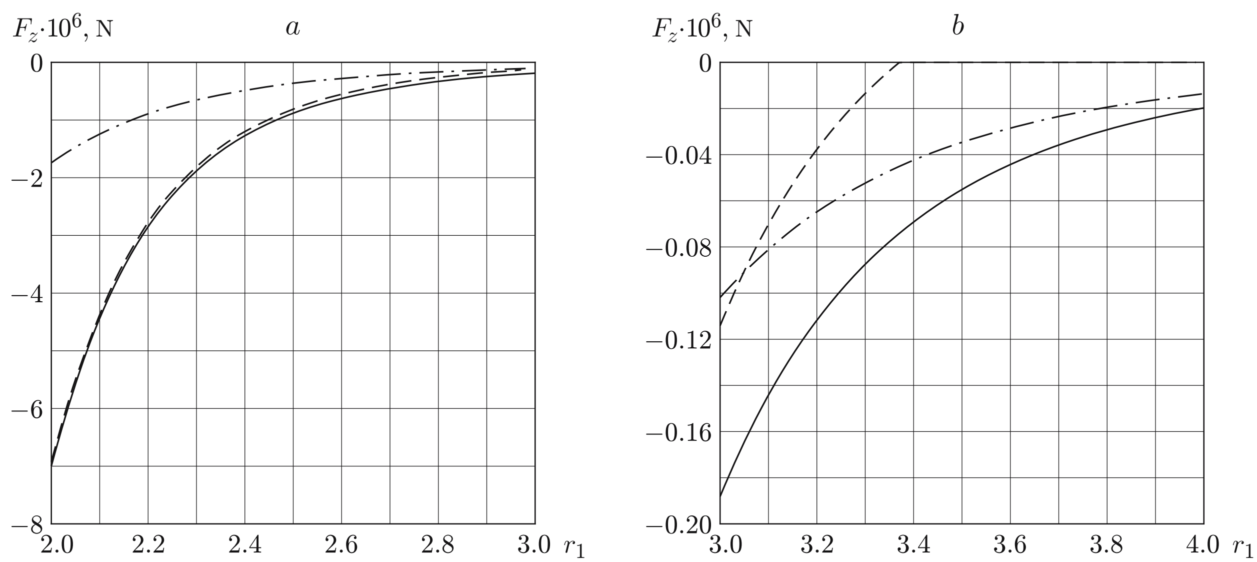

Fig. 3. Dependences of the functions F_{z1} (solid curves), F'_z (dashed curves), and F^a_{z1} (dot-and-dashed curves) on r_1: (a) the drops touch each other; (b) the distance between the drop peripheries is equal to two drop radii.

Let us find Δz from the condition that implies that the mean velocity of the drop at $r_1 = 3$ is equal to the experimental value: $\Delta z = 2V^2_{\text{mean}}/a = -1.24 \cdot 10^{-9}$ m. The drop acceleration at the initial stage of its deceleration with the pressure gradient being neglected is $a' = \pi^2 R_0^2 k_0/m = 1.41 \cdot 10^{-4}$ m/sec^2. The time of drop deceleration is $\Delta t' = -2V_{\text{mean}}/a' = 5.22 \cdot 10^{-3}$ sec. The cycle duration (minus the time during which the drop is at rest, see Introduction) is $\Delta t + \Delta t' = 8.59 \cdot 10^{-3}$ sec.

Conclusions. An algorithm for calculating the force of interaction of two oil drops in a plastic matrix is proposed. The normal component of the stress vector on the drop boundaries is taken from the solution of the problem of the elasticity theory, while the shear component is determined by the plastic properties of the medium

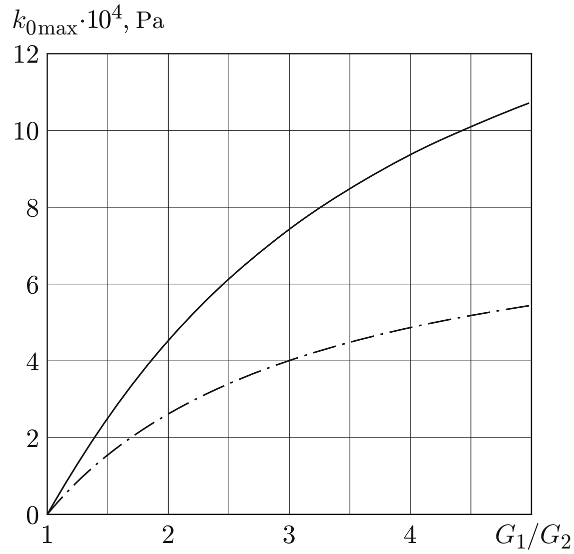

Fig. 4

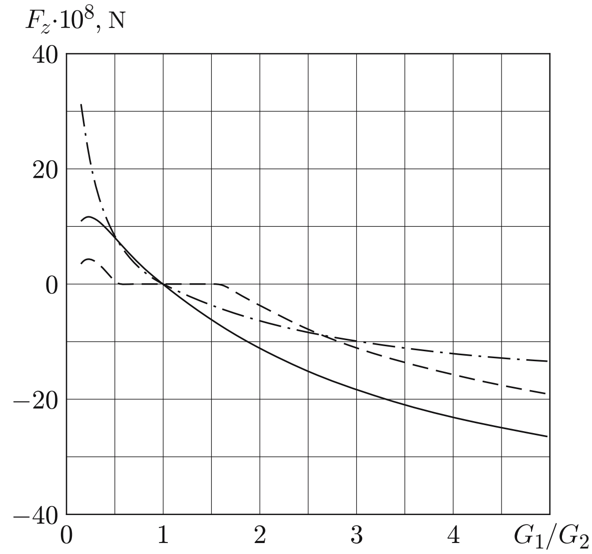

Fig. 5

Fig. 4. Dependence of $k_{0\,\max}$ on G_1/G_2: the solid and dashed curves show the results calculated by the numerical model and by the approximate formula (20), respectively.

Fig. 5. Functions F_{z1} (solid curve), F_z' (dashed curve), and F_{z1}^a (dot-and-dashed curve) versus the ratio G_1/G_2.

(its yield point). Based on calculations performed by the model proposed, the upper estimate of the yield point of the matrix is obtained from the condition that the sign of the force of interaction between the drops corresponds to their mutual attraction at the critical distance where drop approaching is observed in the experiment. With an appropriate choice of Poisson's number, this estimate agrees with experimental data.

The author is grateful to S. V. Stebnovskii, V. V. Pukhnachev, V. V. Kuznetsov, and V. V. Shelukhin for useful discussions of the problem.

This work was supported by the Russian Foundation for Basic Research (Grant No. 07-01-00168) and by the Council on the Grants of the President of the Russian Federation for Supporting the Leading Scientific Schools (Grant No. NSh-2260.2008.1).

REFERENCES

1. S. V. Stebnovskii, "Thermodynamic instability of disperse media isolated from external actions," *J. Appl. Mech. Tech. Phys.*, **40**, No. 3, 407–411 (1999).
2. S. V. Stebnovskii, "Interaction of liquid drops suspended in solutions," *Zh. Tekh. Fiz.*, No. 10, 2177–2180 (1981).
3. S. V. Stebnovskii, "Mechanism of coagulation of disperse elements in media isolated from external actions," *J. Appl. Mech. Tech. Phys.*, **40**, No. 4, 691–696 (1999).
4. A. I. Lur'e, *Spatial Problems of the Elasticity Theory* [in Russian], Gostekhteoretizdat, Moscow (1955).
5. L. D. Landau and E. M. Lifshits, *Theory of Elasticity, Pergamon Press*, Oxford–New York (1970).
6. P. P. Mosolov and V. P. Myasnikov, *Mechanics of Rigid-Plastic Media* [in Russian], Nauka, Moscow (1981).
7. V. V. Shelukhin, "Bingham's fluid model in the stress–velocity variables," *Dokl. Ross. Akad. Nauk*, **377**, No. 4, 455–458 (2001).
8. M. Reiner, *Rheology*, Springer Verlag, Berlin (1958).
9. N. E. Kochin, I. A. Kibel, and N. V. Rose, *Theoretical Hydrodynamics* [in Russian], Fizmatgiz, Moscow (1962).
10. I. K. Kikoin (ed.), *Tables of Physical Quantities: Handbook* [in Russian], Atomizdat, Moscow (1976).
11. R. A. Apakashev and V. V. Pavlov, "Determination of the ultimate strength and shear modulus of water at low velocities of the flow," *Izv. Ross. Akad. Nauk, Mekh. Zhidk. Gaza*, No. 1, 3–7 (1997).
12. V. A. Romanov and S. B. Sapozhnikov, "Experimental determination of viscoelastic characteristics of the fluid," *Izv. Chelyab. Nauch. Tsentra*, No. 4, 69–74 (2002).